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A ONE-DIMENSIONAL MODEL FOR FINITE DEFORMATIONS OF MULTILAYERED RUBBER BEARINGS

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Abstract—A one-dimensional model for multilayered (laminated) rubber bearings is deduced from three-dimensional finite elasticity by means of the imposition of appropriate internal constraints and a suitable choice of the constitutive equation. A closed-form solution is easily obtained in the case of simple compression, so that a load-stretching relation can be established. Small deformations superimposed on finite compression are also studied, and critical compressive loads, superimposed small shear and superimposed small bending are discussed in detail. Finally, to illustrate the significance of the results, some comparisons with previous analyses and experiments are examined.

1. INTRODUCTION

Rubber bearings are widely used in a variety of structural applications. Their use as bridge bearings or as pads in building construction is traditional; moreover, again in the field of civil engineering, they are being increasingly used for the so-called "base isolation", the recent approach to the seismic structural design (see e.g. Kelly, 1991).

The analysis of such structural components is usually based on linear theories (Topaloff, 1964; Courbon *et al.*, 1967), which are inadequate both in principle and in comparison with the experimental results (see e.g. the remarks in the final section of Del Piero and Podio-Guidugli, 1969). Recent research on the subject has been primarily limited to either experimental analysis or finite element analysis, or else specific problems, such as stability and reduction of shear stiffness under increasing axial loads (see, respectively, Roeder and Stanton, 1983; Simo and Kelly, 1984; Stanton *et al.*, 1990, where references in the corresponding area can be found). Attempts to model rubber bearings within the framework of non-linear elasticity were made by Del Piero and Podio-Guidugli (1969) and De Tommasi and Marzano (1988), who obtained explicit solutions of the equilibrium problem for a single rubber layer in simple compression. Furthermore, by using their model for the non-linear elastic deformations of rubber layers, Del Piero and Podio-Guidugli (1970) obtained upper and lower bounds for the collapse load of reinforced bearings.

In this paper we present a one-dimensional model deduced from three-dimensional finite elasticity, concentrating mainly on bearings for isolation base. A typical bearing for such seismic protection is similar, as regards the form, to the traditional bridge bearing; it consists of many thin layers of rubber bonded to horizontal steel plates by means of vulcanization. Due to both the large number of plates and the great flexibility of rubber, the resulting system has high stiffness with respect to vertical loads accompanied by low shear stiffness.

Our present standpoint is different from the usual one. Rather than restricting attention to a typical rubber layer between two successive steel plates, as was done by Del Piero and Podio-Guidugli (1969) and De Tommasi and Marzano (1988), we treat the bearing as a whole, and consider appropriate kinematical assumptions we regard as constitutive prescriptions restricting the class of all possible deformations, i.e. internal constraints. According to the theory of constrained elastic materials, we split the stress tensor in two parts. The one, reactive, is assumed to do no work in any deformation satisfying the constraints; the other, active, is assumed to be both orthogonal to the reaction space and uniquely determined by the deformation. As an explicit example of constitutive function for the active stress, we choose an appropriate form of the Blatz and Ko law for rubber-like materials (Blatz and Ko, 1962). We discuss such a constitutive assumption in Section 2, where two classical homogeneous deformations are also examined.

To derive our one-dimensional model, we consider a special class of plane deformations compatible with the assumed internal constraints; namely, those deformations which preserve distance within each single cross section. Our investigation shows that equilibrium deformations are governed by a boundary-value problem for a non-linear system of differential equations involving three unknown scalar functions of the axial coordinate (Section 3). By considering appropriate boundary conditions, we obtain a closed-form solution in the case of simple compression. As a consequence, we find an exact relation between the stretching parameter and the compressive force, i.e. the load-stretching relation. Closed-form solutions are not in sight for other types of loadings. Our work on numerical solutions has not been completed yet; we plan to present it in a future paper.

Here we turn to the study of small deformations superimposed on finite compression, using a standard linearization procedure (Section 4). We indicate how to obtain critical loads in simple compression, and, with reference to the lowest critical load, we examine the dependence on geometrical parameters (Subsection 4.1). For superimposed deformations, being either small shear (Subsection 4.2) or small bending (Subsection 4.3), we also perform the explicit calculation of the corresponding stiffness. Interestingly, for shear, our results predict that the *transverse stiffness* is specified by a decreasing function of the compressive force, just as several experimental results and semi-empirical formulae deduce. Conversely, the *bending stiffness* increases with increasing compressive load (until its maximum value), and this theoretical result is also confirmed by experiments.

Finally, specific applications of our work in modelling the behavior of multilayered rubber bearings are illustrated in Section 5. Here, we discuss the significance of our results by means of a detailed comparison with previous analysis and experiments.

2. CONSTITUTIVE ASSUMPTIONS

As a first step, we now introduce the constitutive class on which our approach is based.

Let $\{\mathbf{O}; \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be a Cartesian coordinate frame, and let (X_1, X_2, X_3) be the coordinates of a point $\mathbf{X}$. Let us consider the parallelepiped $\mathscr{B}$ with edge lengths $2A$, $2B$, L having the basis $\mathscr{A} :=]-A, A[\times]-B, B[$ in the plane $X_3 = 0$ (Fig. 1). As alluded to in the Introduction, we treat the bearing as an internally constrained elastic block, identified with the region $\mathscr{B} = \mathscr{A} \times]0, L[$ it occupies in an undistorted reference configuration. We

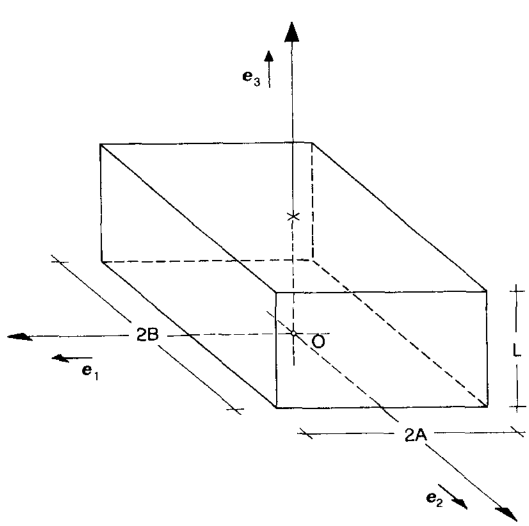

Fig. 1.

suppose that the steel reinforcing layers are parallel to the (X_1, X_2) plane, and we call $\mathscr{A}(X_3)$ the typical cross section of $\mathscr{B}$.

A deformation of $\mathscr{B}$ is a smooth mapping $\mathbf{f}\colon \mathscr{B} \ni \mathbf{X} \mapsto \mathbf{x} = \mathbf{f}(\mathbf{X}) \in \mathbb{R}^3$ with $\det \mathbf{F} > 0$, where

$$\mathbf{F} = \nabla \mathbf{f} \tag{1}$$

is the deformation gradient.

We impose the following internal constraints:

(h_1) orthogonality preserving with respect to direction pair $(\mathbf{e}_2, \mathbf{e}_3)$;
(h_2) inextensibility in the directions $\mathbf{e}_1$, $\mathbf{e}_2$, and orthogonality preserving with respect to the direction pair $(\mathbf{e}_1, \mathbf{e}_2)$.

Thus, for each $\mathbf{X} \in \mathscr{B}$, the set of all admissible deformation gradients $\mathbf{F}$ from the reference placement is a *constraint manifold* described by the restrictions:

$$\mathbf{F}\mathbf{e}_2 \cdot \mathbf{F}\mathbf{e}_3 = 0 \quad \text{and} \quad \mathbf{F}\mathbf{e}_i \cdot \mathbf{F}\mathbf{e}_j - \delta_{ij} = 0 \quad \text{for } i,j = 1,2 \tag{2}$$

(δ_{ij} is the *Kronecker delta*).

Notice that the second part of (2) implies inextensibility in all directions coplanar with the two directions $\mathbf{e}_1$, $\mathbf{e}_2$, that is a local (in-plane) rigidity, an assumption justified by the very large number of the steel plates vulcanized inside the rubber matrix. The imposition of (h_1), which is described by (2) part 1, is motivated by our procedure of constructing the one-dimensional model from the three-dimensional theory.

According to the theory of constrained elastic materials,* we now lay down a constitutive prescription for the Cauchy stress tensor $\mathbf{T}$ as follows. We split $\mathbf{T}$ into an active part $\mathbf{T}^{\mathrm{A}}$ and a reactive part $\mathbf{T}^{\mathrm{R}}$:

$$\mathbf{T} = \mathbf{T}^{\mathrm{A}} + \mathbf{T}^{\mathrm{R}}. \tag{3}$$

Further, we stipulate that the reactive stress $\mathbf{T}^{\mathrm{R}}$ does no work for any deformation satisfying the assumed constraints (2). As an immediate consequence, we have that $\mathbf{T}^{\mathrm{R}}$ can be expressed in the form

$$\mathbf{T}^{\mathrm{R}} = \rho_1 \mathbf{a}_1 \otimes \mathbf{a}_1 + \rho_2 \mathbf{a}_2 \otimes \mathbf{a}_2 + \rho_3(\mathbf{a}_1 \otimes \mathbf{a}_2 + \mathbf{a}_2 \otimes \mathbf{a}_1) + \rho_4(\mathbf{a}_2 \otimes \mathbf{a}_3 + \mathbf{a}_3 \otimes \mathbf{a}_2), \tag{4}$$

where ρ_i $(i = 1, \ldots, 4)$ denote arbitrary scalars, and

$$\mathbf{a}_1 := \mathbf{F}\mathbf{e}_1, \quad \mathbf{a}_2 := \mathbf{F}\mathbf{e}_2, \quad \mathbf{a}_3 := \mathbf{F}\mathbf{e}_1 \times \mathbf{F}\mathbf{e}_2 \tag{5}$$

are three mutually orthogonal unit vectors (for any admissible $\mathbf{F}$).

Finally, the active stresses $\mathbf{T}^{\mathrm{A}}$ are assumed to be orthogonal to the reactive stresses $\mathbf{T}^{\mathrm{R}}$. Under the imposed constraints, this is equivalent to requiring that the active response function be a mapping $\mathbf{F} \mapsto \mathbf{T}^{\mathrm{A}} = \hat{\mathbf{T}}^{\mathrm{A}}(\mathbf{F})$ of the form:

$$\hat{\mathbf{T}}^{\mathrm{A}}(\mathbf{F}) = \hat{\alpha}_1(\mathbf{F})\mathbf{a}_3 \otimes \mathbf{a}_3 + \hat{\alpha}_2(\mathbf{F})(\mathbf{a}_1 \otimes \mathbf{a}_3 + \mathbf{a}_3 \otimes \mathbf{a}_1), \tag{6}$$

where $\mathbf{a}_1$, $\mathbf{a}_3$ are given by (5), and where $\hat{\alpha}_1$, $\hat{\alpha}_2$ are scalar functions (with obvious domain). It is important to note that the freedom in the choice of such $\hat{\alpha}_1$, $\hat{\alpha}_2$ is reduced by the requirement that the mapping $\hat{\mathbf{T}}^{\mathrm{A}}$ is *frame-indifferent*. Indeed, it is not difficult to show that frame-indifference corresponds to the requirement that:

* We follow the theory of constrained materials in the form proposed by Gurtin and Podio-Guidugli (1973). We also refer the reader to the recent article by Podio-Guidugli (1990).

$$\hat{\alpha}_i(\mathbf{RF}) = \hat{\alpha}_i(\mathbf{F}) \quad (i = 1,2), \tag{7}$$

for every proper orthogonal tensor **R** and every admissible deformation gradient **F**.

In later calculations we shall use the following explicit expressions for the response functions $\hat{\alpha}_1$, $\hat{\alpha}_2$:

$$\begin{aligned} \hat{\alpha}_1(\mathbf{F}) &= \mu[\det \mathbf{F} - (\det \mathbf{F})^{-(\sigma+1)}] + \lambda(\det \mathbf{F})(|\mathbf{Fe}_3|^2 - 1), \\ \hat{\alpha}_2(\mathbf{F}) &= [\mu + \lambda(|\mathbf{Fe}_3|^2 - 1)](\mathbf{Fe}_1 \cdot \mathbf{Fe}_3), \end{aligned} \tag{8}$$

where μ, λ and σ are *material moduli* satisfying the inequalities

$$\mu > \lambda > 0, \quad \sigma > 0. \tag{9}$$

Clearly, constitutive prescriptions (8) agree with the frame-indifference requirement (7).

Remark 1. For $\hat{\alpha}_1$ and $\hat{\alpha}_2$ chosen as in (8), (6) specifies the active stress for hyperelastic materials, constrained so as requested by (2) and having the following stored energy density $\hat{\sigma}$:

$$\hat{\sigma} : \mathbf{F} \mapsto \hat{\sigma}(\mathbf{F}) := \tfrac{1}{2}\mu\left[\|\mathbf{F}\|^2 - 3 + \frac{2}{\sigma}((\det \mathbf{F})^{-\sigma} - 1)\right] + \frac{\lambda}{4}(|\mathbf{Fe}_3|^2 - 1)^2. \tag{10}$$

Indeed, by considering the orthogonal projection of the values of the mapping

$$\mathbf{F} \mapsto (\det \mathbf{F})^{-1} D\hat{\sigma}(\mathbf{F})\mathbf{F}^{\mathrm{T}} \tag{11}$$

on the active stress space, we find (6) and (8). As remarked by Vianello (1990), $D\hat{\sigma}$ in the last formula denotes the surface gradient of $\hat{\sigma}$ on the constraint manifold.

Remark 2. For $\lambda = 0$, the mapping (10) reduces to the stored energy density considered by Blatz and Ko (1962) to model the mechanical behavior of rubbers. However, the addition of the term $\lambda(|\mathbf{Fe}_3|^2 - 1)^2/4$ is important to avoid the unreal behavior of Blatz and Ko material, which tends to "soften" in tension (see Burgess and Levinson, 1972).

We now discuss the behavior of the above constitutive relations limiting attention to two classical homogeneous deformations; we shall use components where convenient.

(a) *Extension*. Consider an extension of amount k (with $k > 0$) in the direction $\mathbf{e}_3$, so that

$$\mathbf{F} = \mathbf{I} + (k-1)\mathbf{e}_3 \otimes \mathbf{e}_3. \tag{12}$$

If we assume

$$\rho_i = 0 \quad (i = 1, \ldots, 4) \tag{13}$$

for the parameters which determine the reaction stress $\mathbf{T}^{\mathrm{R}}$ [see eqn (4)], we obtain by (3), (6) and (8) that the corresponding stress $\mathbf{T}$ is a pure tension in the direction $\mathbf{e}_3$:

$$\mathbf{T} = T_{33}\mathbf{e}_3 \otimes \mathbf{e}_3, \tag{14}$$

with

$$T_{33} = \mu(k - k^{-(\sigma+1)}) + \lambda(k^3 - k). \tag{15}$$

It is worth noting that, under inequalities (9), the normal stress T_{33} is a strictly increasing

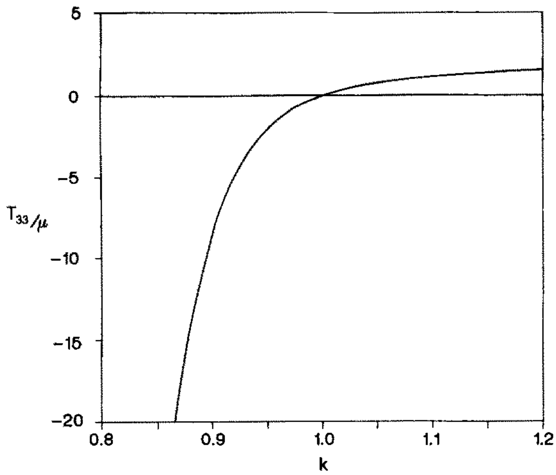

Fig. 2. The normal stress T_{33}/μ as a function of k ($\sigma = 20$, $\lambda/\mu = 0.8$).

function of k (Fig. 2). In particular, $T_{33} \to -\infty$ as $k \to 0$, and $T_{33} \to \infty$ as $k \to \infty$; in other words, the stress tends to $\mp\infty$ as the length of the specimen tends to zero or infinity, respectively. Figure 2 and the following are obtained for $\sigma = 20$ and $\lambda/\mu = 0.8$, a suitable choice for laminated elastomeric bearings.

(*b*) *Simple shear.* Let

$$\mathbf{F} = \mathbf{I} + \gamma(\mathbf{e}_1 \otimes \mathbf{e}_3) \tag{16}$$

be a simple shear, with axis $\mathbf{e}_1$ and shearing strain γ. Again in view of (3), (4), (6) and (8), a simple calculation shows that the components $T_{13} = \mathbf{T}\mathbf{e}_3 \cdot \mathbf{e}_1$ and $T_{33} = \mathbf{T}\mathbf{e}_3 \cdot \mathbf{e}_3$ have the form

$$T_{13} = \mu\gamma + \lambda\gamma^3 \quad \text{and} \quad T_{33} = \lambda\gamma^2. \tag{17}$$

Thus, both the shear stress T_{13} and the normal stress T_{33} are uniquely determined by the deformation. T_{13} is an odd function of the shear strain γ; moreover, for T_{33} to vanish λ would have to be zero. Constitutive restrictions (9) imply that the *generalized shear modulus* T_{13}/γ is increasing with γ.

3. DERIVATION OF THE MODEL

In this section we derive a one-dimensional equilibrium problem for our elastic solid $\mathscr{B}$. We begin by imposing the momentum balance laws for any part of $\mathscr{B}$ which is bounded by two planes orthogonal to direction $\mathbf{e}_3$, namely, for parts $\mathscr{P}$ having the form:

$$\mathscr{P} = \mathscr{A} \times]d_1, d_2[, \tag{18}$$

where d_1, d_2 (with $d_1 < d_2$) can be arbitrarily chosen in $[0, L]$. Then, in absence of body force, the balance equations take the form

$$\int_{\partial\mathscr{P}} \mathbf{S}\mathbf{n}\, \mathrm{d}A = \mathbf{0},$$

$$\int_{\partial\mathscr{P}} (\mathbf{x} - \mathbf{O}) \times \mathbf{S}\mathbf{n}\, \mathrm{d}A = \mathbf{0} \quad \text{for any } \mathscr{P}, \tag{19}$$

where $\mathbf{n}$ is the outward unit normal field on $\partial\mathscr{P}$, and $\mathbf{S}$ denotes the Piola–Kirchhoff stress field depending on the Cauchy stress $\mathbf{T}$ and the deformation gradient $\mathbf{F}$ by means of the well known relation

$$\mathbf{S} = (\det \mathbf{F})\mathbf{T}\mathbf{F}^{-\mathrm{T}}. \tag{20}$$

We define the force $\mathbf{R}(X_3)$ and the moment $\mathbf{M}(X_3)$ (about $\mathbf{O}$) on $\mathscr{A}(X_3)$ by

$$\mathbf{R}(X_3) := \int_{\mathscr{A}(X_3)} \mathbf{S}\mathbf{e}_3 \, \mathrm{d}A, \quad \mathbf{M}(X_3) := \int_{\mathscr{A}(X_3)} (\mathbf{x}-\mathbf{O}) \times \mathbf{S}\mathbf{e}_3 \, \mathrm{d}A. \tag{21}$$

By (19), using standard arguments, we arrive at the differential equations

$$\frac{\mathrm{d}\mathbf{R}}{\mathrm{d}X_3} + \mathbf{r} = \mathbf{0}, \quad \frac{\mathrm{d}\mathbf{M}}{\mathrm{d}X_3} + \mathbf{m} = \mathbf{0} \quad \text{in }]0, L[, \tag{22}$$

where

$$\mathbf{r}(X_3) := \int_{\partial\mathscr{A}(X_3)} \mathbf{S}\mathbf{n} \, \mathrm{d}s \quad \text{and} \quad \mathbf{m}(X_3) := \int_{\partial\mathscr{A}(X_3)} (\mathbf{x}-\mathbf{O}) \times \mathbf{S}\mathbf{n} \, \mathrm{d}s \tag{23}$$

are, respectively, the force and moment of the traction on the boundary of $\mathscr{A}(X_3)$.

If, as we assume here, the lateral surface of $\mathscr{B}$ is traction-free, then $\mathbf{S}$ must satisfy the boundary condition

$$\mathbf{S}\mathbf{n} = \mathbf{0} \quad \text{on} \quad \partial\mathscr{A} \times]0, L[. \tag{24}$$

So that, by (23),

$$\mathbf{r} = \mathbf{0}, \quad \mathbf{m} = \mathbf{0} \quad \text{in }]0, L[, \tag{25}$$

and the equilibrium equations (22) can be written as

$$\frac{\mathrm{d}\mathbf{R}}{\mathrm{d}X_3} = \mathbf{0}, \quad \frac{\mathrm{d}\mathbf{M}}{\mathrm{d}X_3} = \mathbf{0} \quad \text{in }]0, L[. \tag{26}$$

To derive our one-dimensional model, we study a special class of plane deformations compatible with the assumed internal constraints (2). Precisely, we consider deformations of the form:

$$\begin{cases} x_1 = g(X_3) + X_1 \cos \nu(X_3) \\ x_2 = X_2 \\ x_3 = h(X_3) + X_1 \sin \nu(X_3), \end{cases} \tag{27}$$

where (x_1, x_2, x_3) denote the coordinates of the point $\mathbf{x} = \mathbf{f}(\mathbf{X})$, and g, h, ν are arbitrary functions. The corresponding deformation gradient has the form:

$$[\mathbf{F}] = \begin{bmatrix} \cos\nu & 0 & g' - X_1\nu' \sin\nu \\ 0 & 1 & 0 \\ \sin\nu & 0 & h' + X_1\nu' \cos\nu \end{bmatrix} \tag{28}$$

(the prime denotes differentiation with respect to X_3), so that, as is immediately seen, each of the imposed constraints (2) is satisfied.

Thus, for the above equilibrium problem, we now make explicit the selected class of deformations and constitutive prescriptions. We begin by observing that, by (3) and (20), the Piola–Kirchhoff stress $\mathbf{S}$ can be decomposed in a reactive part $\mathbf{S}^R$ and an active part $\mathbf{S}^A$:

$$\mathbf{S} = \mathbf{S}^R + \mathbf{S}^A, \tag{29}$$

with $\mathbf{S}^R := (\det \mathbf{F})\mathbf{T}^R\mathbf{F}^{-T}$ and $\mathbf{S}^A := (\det \mathbf{F})\mathbf{T}^A\mathbf{F}^{-T}$. Thus, when $\mathbf{F}$ is chosen as in (28), $\mathbf{T}^R$ as in (4) and $\mathbf{T}^A$ as in (6), we obtain

$$[\mathbf{S}^R] = \begin{bmatrix} J\rho_1 \cos v & J(\rho_3 \cos v - \rho_4 \sin v) & 0 \\ J\rho_3 - (h' \sin v + g' \cos v)\rho_4 & J\rho_2 & \rho_4 \\ J\rho_1 \sin v & J(\rho_3 \sin v + \rho_4 \cos v) & 0 \end{bmatrix},$$

$$[\mathbf{S}^A] = \begin{bmatrix} -(h' \sin v + g' \cos v)(\hat{\alpha}_1(\mathbf{F}) \sin v & 0 & -\hat{\alpha}_1(\mathbf{F}) \sin v + \hat{\alpha}_2(\mathbf{F}) \cos v \\ \quad + \hat{\alpha}_2(\mathbf{F}) \cos v) - J\hat{\alpha}_2(\mathbf{F}) \sin v & & \\ 0 & 0 & 0 \\ -(h' \sin v + g' \cos v)(\hat{\alpha}_1(\mathbf{F}) \cos v & 0 & \hat{\alpha}_1(\mathbf{F}) \cos v + \hat{\alpha}_2(\mathbf{F}) \sin v \\ \quad + \hat{\alpha}_2(\mathbf{F}) \sin v) + J\hat{\alpha}_2(\mathbf{F}) \cos v & & \end{bmatrix}, \tag{30}$$

where we have set

$$J := \det \mathbf{F} = h' \cos v - g' \sin v + X_1 v', \tag{31}$$

and where, in view of (8),

$$\begin{aligned} \hat{\alpha}_1(\mathbf{F}) &= \mu(J - J^{-(\sigma+1)}) + \lambda J[g'^2 + h'^2 - 1 + X_1^2 v'^2 + 2X_1 v'(h' \cos v - g' \sin v)], \\ \hat{\alpha}_2(\mathbf{F}) &= (g' \cos v + h' \sin v)\{\mu + \lambda[g'^2 + h'^2 - 1 + X_1^2 v'^2 + 2X_1 v'(h' \cos v - g' \sin v)]\}. \end{aligned} \tag{32}$$

Throughout, because of the symmetry of the problem with respect to the plane $X_2 = 0$, we assume for the reaction fields ρ_i:

$$\begin{aligned} \rho_i(X_1, X_2, X_3) &= \rho_i(X_1, -X_2, X_3), \quad i = 1, 2, \\ \rho_i(X_1, X_2, X_3) &= -\rho_i(X_1, -X_2, X_3), \quad i = 3, 4. \end{aligned} \tag{33}$$

So, expressions for the force $\mathbf{R}$ and moment $\mathbf{M}$ on the typical section $\mathscr{A}(X_3)$ follow from definitions (21) with the use of (27), (29), (30) and (32). They are:

$$\mathbf{R}(X_3) = T(X_3)\mathbf{e}_1 + N(X_3)\mathbf{e}_3 \quad \text{and} \quad \mathbf{M}(X_3) = M(X_3)\mathbf{e}_2, \tag{34}$$

where

$$N(X_3) = \int_{\mathscr{A}(X_3)} (\hat{\alpha}_1(\mathbf{F}) \cos v + \hat{\alpha}_2(\mathbf{F}) \sin v) \, dA = 4AB\left[\mu + \lambda(g'^2 + h'^2 - 1) + \frac{\lambda}{3} A^2 v'^2\right] h'$$
$$+ \left[\frac{8}{3}\lambda A^3 B v'^2 (h' \cos v - g' \sin v) - 2\mu B \int_{-A}^{A} J^{-(\sigma+1)} \, dX_1\right] \cos v,$$

$$T(X_3) = \int_{\mathscr{A}(X_3)} (\hat{\alpha}_2(\mathbf{F})\cos v - \hat{\alpha}_1(\mathbf{F})\sin v)\,\mathrm{d}A = 4AB\left[\mu + \lambda(g'^2 + h'^2 - 1) + \frac{\lambda}{3}A^2 v'^2\right] g'$$

$$- \left[\frac{8}{3}\lambda A^3 B v'^2 (h'\cos v - g'\sin v) - 2\mu B \int_{-A}^{A} J^{-(\sigma+1)}\,\mathrm{d}X_1\right]\sin v,$$

$$M(X_3) = \int_{\mathscr{A}(X_3)} [\hat{\alpha}_2(\mathbf{F})(h\cos v - g\sin v) - \hat{\alpha}_1(\mathbf{F})(h\sin v + g\cos v + X_1)]\,\mathrm{d}A$$

$$= hT - gN - \frac{8}{3}\lambda A^3 B (h'\cos v - g'\sin v)^2 v'$$

$$- \frac{4}{3}A^3 B[\mu + \lambda(g'^2 + h'^2 - 1)]v' - \frac{4}{5}\lambda A^5 B v'^3 + 2\mu B \int_{-A}^{A} X_1 J^{-(\sigma+1)}\,\mathrm{d}X_1. \quad (35)$$

Similar calculations for the force $\mathbf{r}$ and the moment $\mathbf{m}$ defined by (23) yield

$$\mathbf{r}(X_3) = \left\{\cos vI_1 - \sin vI_2 + 2B[S_{11}^A]_{X_1=-A}^{X_1=A}\right\}\mathbf{e}_1 + \left\{\sin vI_1 + \cos vI_2 + 2B[S_{31}^A]_{X_1=-A}^{X_1=A}\right\}\mathbf{e}_3,$$

$$\mathbf{m}(X_3) = \Big\{(h\cos v - g\sin v)I_1 - (h\sin v + g\cos v)I_2 - I_3$$

$$+ 2B[S_{11}^A(h + X_1\sin v) - S_{31}^A(g + X_1\cos v)]_{X_1=-A}^{X_1=A}\Big\}\mathbf{e}_2, \quad (36)$$

where we have set

$$I_1 := \int_{-B}^{B} [J\rho_1]_{X_1=-A}^{X_1=A}\,\mathrm{d}X_2 + 2\int_{-A}^{A} (J\rho_3)|_{X_2=B}\,\mathrm{d}X_1,$$

$$I_2 := 2\int_{-A}^{A} (J\rho_4)|_{X_2=B}\,\mathrm{d}X_1, \quad I_3 := 2\int_{-A}^{A} X_1(J\rho_4)|_{X_2=B}\,\mathrm{d}X_1. \quad (37)$$

Equations (34) and (35) make evident the "active nature" of $\mathbf{R}$ and $\mathbf{M}$ which are uniquely determined by the deformation $\mathbf{f}$. Conversely, as shown by (36) and (37), $\mathbf{r}$ and $\mathbf{m}$ are determined by $\mathbf{f}$ only to within the arbitrary reactive fields ρ_1, ρ_3, ρ_4. It is not difficult to show that the boundary conditions (25) are always satisfied when $\mathbf{r}$ and $\mathbf{m}$ are given by (36) and (37). Indeed, by virtue of the arbitrariness of ρ_1, ρ_3, ρ_4, it is easily seen that $I_1(X_3)$, $I_2(X_3)$ and $I_3(X_3)$ can assume any set of values for each $X_3 \in]0, L[$. In particular, we can choose $I_1(X_3)$, $I_2(X_3)$ and $I_3(X_3)$ such that all components of $\mathbf{r}(X_3)$ and $\mathbf{m}(X_3)$ vanish for every $X_3 \in]0, L[$.

Combining (26) and (34), the equilibrium equations can be written in scalar form as:

$$\frac{\mathrm{d}N}{\mathrm{d}X_3} = 0, \quad \frac{\mathrm{d}T}{\mathrm{d}X_3} = 0, \quad \frac{\mathrm{d}M}{\mathrm{d}X_3} = 0 \quad \text{in }]0, L[. \quad (38)$$

In conclusion, the equilibrium problem can be stated as follows: for $\mathbf{f}$ given by (27), find the functions g, h and v which satisfy (38), (35). Of course, boundary conditions which prescribe the deformation or the total traction on the end faces $X_3 = 0$ and $X_3 = L$ must be added to the system of ordinary differential equations (38).

4. SMALL DEFORMATIONS SUPERIMPOSED ON FINITE COMPRESSION

Firstly, we consider a finite compression. Let kL, with $0 < k < 1$, be the length of $\mathscr{B}$ after deformation. If we assume the boundary conditions

$$\begin{aligned} h(0) &= 0, \quad g(0) = 0, \quad v(0) = 0, \\ h(L) &= kL, \quad g(L) = 0, \quad v(L) = 0, \end{aligned} \tag{39}$$

then it is easily seen that

$$g(X_3) = 0, \quad h(X_3) = kX_3, \quad v(X_3) = 0 \tag{40}$$

is a solution of the boundary value problem defined by (35), (38) and (39). In other words, we have an extension of amount k in the direction $\mathbf{e}_3$. Moreover, again by (35), a trivial computation shows that $T = 0$, $M = 0$, and

$$N(X_3) = 4ABT_{33} =: N_0 \tag{41}$$

with T_{33} given by (15). Thus, N is independent of X_3, and the function $k \mapsto N_0(k)$ defined by (41) and (15) can be interpreted as the relation between the *stretching parameter* k and the normal force N_0 acting on the basis. To within the area of the basis, such a relation is shown in Fig. 2.

As there is little hope of solving the non-linear differential system (35), (38) in closed-form when the boundary conditions differ from (39), we confine our attention to small deformations superimposed on the finite compression defined by (40). More precisely, we now set formally

$$\begin{cases} g(X_3) = \varepsilon\tilde{g}(X_3) \\ h(X_3) = kX_3 + \varepsilon\tilde{h}(X_3), \\ v(X_3) = \varepsilon\tilde{v}(X_3) \end{cases} \tag{42}$$

where ε denotes a smallness parameter. Then, to within terms of order $O(\varepsilon^2)$ as $\varepsilon \to 0$, eqns (35) yield

$$\begin{aligned} N(X_3) &= N_0(k) + \varepsilon a(k)\tilde{h}'(X_3), \\ T(X_3) &= \varepsilon b(k)\tilde{g}'(X_3) + \varepsilon c(k)\tilde{v}(X_3), \\ M(X_3) &= kX_3T(X_3) - \varepsilon N_0(k)\tilde{g}(X_3) - \varepsilon\frac{A^2}{3}a(k)\tilde{v}'(X_3), \end{aligned} \tag{43}$$

where

$$\frac{N_0(k)}{4AB\mu} = k - k^{-(\sigma+1)} + \frac{\lambda}{\mu}(k^3 - k), \tag{44}$$

and

$$\begin{aligned} \frac{a(k)}{4AB\mu} &= 1 + (\sigma+1)k^{-(\sigma+2)} + \frac{\lambda}{\mu}(3k^2 - 1), \\ \frac{b(k)}{4AB\mu} &= 1 + \frac{\lambda}{\mu}(k^2 - 1), \\ \frac{c(k)}{4AB\mu} &= k^{-(\sigma+1)}. \end{aligned} \tag{45}$$

A simple computation shows that the inequalities (9) imply

$$N_0(k) < 0 \quad \text{and} \quad a(k) > 0, b(k) > 0, c(k) > 0 \tag{46}$$

for every $k \in]0, 1[$.

For the remainder of this section, we write $\bar{g}$, $\bar{h}$, $\bar{v}$ in place of $\varepsilon\tilde{g}$, $\varepsilon\tilde{h}$, $\varepsilon\tilde{v}$, respectively. Substitution of (43) into (38) yields

$$\begin{cases} a(k)\bar{h}'' = 0 & \\ b(k)\bar{g}'' + c(k)\bar{v}' = 0 & \text{in }]0, L[, \\ k[b(k)\bar{g}' + c(k)\bar{v}] - N_0(k)\bar{g}' - \frac{A^2}{3}a(k)\bar{v}'' = 0 & \end{cases} \tag{47}$$

a linear differential system whose solution, for $0 < k < 1$, has the form:

$$\begin{aligned} \bar{h}(X_3) &= C_1X_3 + C_2, \\ \bar{g}(X_3) &= C_3 \sin(d(k)X_3) + C_4 \cos(d(k)X_3) + C_5kX_3 + C_6, \\ \bar{v}(X_3) &= -\frac{b(k)d(k)}{c(k)}[C_3 \cos(d(k)X_3) - C_4 \sin(d(k)X_3)] - C_5, \end{aligned} \tag{48}$$

where $C_1, \ldots, C_6$ are arbitrary constants, and the function $]0, 1[\ni k \mapsto d(k)$ is defined by

$$d(k) := \frac{1}{A}\sqrt{\frac{-3N_0(k)c(k)}{a(k)b(k)}}. \tag{49}$$

Notice that, in view of (44) and (45), $k \mapsto d(k)$ is continuous and strictly decreasing in $]0, 1[$; further, $d \to +\infty$ as $k \to 0$, and $d \to 0$ as $k \to 1$.

4.1. *Critical loads in simple compression*

Consider the boundary-value problem obtained by adjoining to (47) the following homogeneous boundary conditions:

$$\bar{h}(0) = \bar{h}(L) = 0, \quad \bar{g}(0) = \bar{g}(L) = 0, \quad \bar{v}(0) = \bar{v}(L) = 0. \tag{50}$$

Our next step will be to find the *critical* values of the stretching parameter k, i.e. the values of k for which the above boundary-value problem admits non-trivial solutions. We do this by combining (48) and (50). Then, for the resulting linear system in the unknown $(C_1, \ldots, C_6)$, the coefficients matrix assumes the form

$$\left[\begin{array}{cc|cccc} 0 & 1 & & & & \\ L & 1 & & 0 & & \\ \hline & & 0 & 1 & 0 & 1 \\ & & \sin(d(k)L) & \cos(d(k)L) & kL & 1 \\ & 0 & -\frac{b(k)d(k)}{c(k)} & 0 & -1 & 0 \\ & & -\frac{b(k)d(k)}{c(k)}\cos(d(k)L) & \frac{b(k)d(k)}{c(k)}\sin(d(k)L) & -1 & 0 \end{array}\right]. \tag{51}$$

For the system to have a non-trivial solution, we must require the vanishing of the determinant of (51). As a simple calculation shows, such a requirement is expressed by the equation

$$\frac{b(k)d(k)}{c(k)}\sin^2\frac{d(k)L}{2}\left(2-\frac{b(k)d(k)}{c(k)}kL\cot\frac{d(k)L}{2}\right)=0, \tag{52}$$

with $k\in]0,1[$. Therefore, the roots of equation (52) which belong to $]0,1[$ are the desired values of k. Clearly, compressive critical loads can be obtained by evaluating the expression (44) at such values of k.

To evaluate the greatest critical value of k we observe that, in view of (46) and (49), (52) is equivalent to either

$$d(k)L = 2n\pi \quad (n \text{ integer}) \tag{53a}$$

or

$$\frac{b(k)k}{c(k)}\frac{d(k)L}{2}\cot\frac{d(k)L}{2}=1. \tag{53b}$$

Let k_{cr}^1 denote the critical value given by (53a) with $n = 1$, so that

$$d(k_{cr}^1)L = 2\pi. \tag{54}$$

Our next step will be to show that k_{cr}^1 is just the greatest critical value. Clearly, since the function $k\mapsto d(k)$ is strictly decreasing in $]0,1[$, k_{cr}^1 is the greatest critical value satisfying (53a). Thus, we have only to show that there is no value of $k\in]k_{cr}^1,1[$ satisfying (53b). Indeed, by (45),

$$\frac{b(k)k}{c(k)}<1 \tag{55}$$

for every $k\in]0,1[$, so that (53b) implies

$$\frac{d(k)L}{2}\cot\frac{d(k)L}{2}>1. \tag{56}$$

But, as is easy to see, this inequality cannot hold if $k\in]k_{cr}^1,1[$, since properties of the function d in conjunction with (54) require that $(d(k)L/2)\in]0,\pi[$ when $k\in]k_{cr}^1,1[$. We therefore conclude that k_{cr}^1 is the desired critical value. Of course, the corresponding critical load $N_0(k_{cr}^1)$ is the lowest critical load in absolute value, since the function $k\mapsto N_0(k)$ defined by (44) is negative and strictly increasing in $]0,1[$.

In our structural applications, we call the ratio L/A the *slenderness* of the bearing. Clearly, k_{cr}^1 depends on the slenderness. Specifically, in view of (49), (54) defines the function

$$L/A\mapsto k_{cr}^1\in]0,1[\tag{57}$$

which is increasing in $]0,+\infty[$ since d is decreasing. Similar arguments show that $|N_0(k_{cr}^1)|$, the absolute value of the smallest critical load, is a decreasing function of L/A [see curve (a) in Fig. 3, where we use the dimensionless parameter $|N_0|/4AB\mu$ in place of $|N_0|$].

4.2. *Superimposed small shear*

Here the boundary conditions take the form

$$\bar{g}(0)=0,\quad \bar{h}(0)=0,\quad \bar{v}(0)=0\quad\text{and}\quad \bar{g}(L)=s,\quad \bar{h}(L)=0,\quad \bar{v}(L)=0, \tag{58}$$

with s a small parameter. In other words, from the state of finite compression, the bases are constrained to undergo a rigid relative displacement of amount s in the direction of X_1-axis. Substitution of (48) into (58) leads to a non-homogeneous system of linear equation

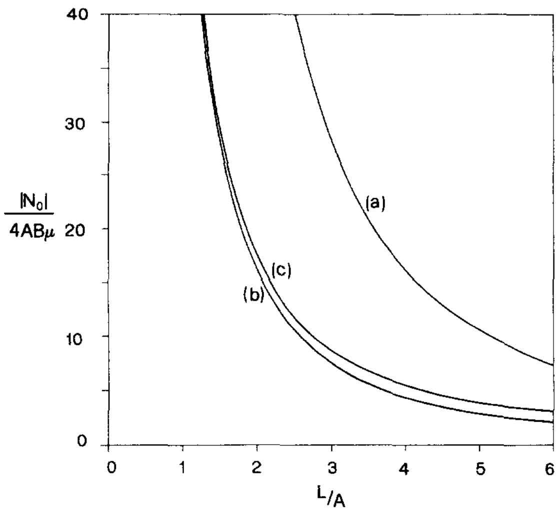

Fig. 3. Critical load $|N_0|/4AB\mu$ as a function of L/A. (a) Simple compression; (b) superimposed small shear; (c) superimposed small bending ($\sigma = 20$, $\lambda/\mu = 0.8$).

in the unknowns $(C_1, \ldots, C_6)$ whose coefficients matrix is again (51). Thus, the boundary-value problem defined by (47) and (58) has a unique solution when (51) is non-singular; so it is, in particular, for every $k \in]k^1_{cr}, 1[$, with k^1_{cr} defined by (54).

A routine calculation yields

$$\begin{aligned} C_1 &= C_2 = 0, \\ C_3 &= -Q, \\ C_4 &= -C_6 = \tan \frac{d(k)L}{2} Q, \\ C_5 &= \frac{b(k)d(k)}{c(k)} Q, \end{aligned} \tag{59}$$

where we have set

$$Q = \frac{s}{kL \dfrac{b(k)d(k)}{c(k)} - 2 \tan \dfrac{d(k)L}{2}}. \tag{60}$$

Finally, (43) gives the following expression for the shear force T:

$$T = \frac{N_0(k)}{kL - 2 \dfrac{c(k)}{b(k)d(k)} \tan \dfrac{d(k)L}{2}} s. \tag{61}$$

In order to apply our results to cases of technical interest, we introduce the (dimensionless) *transverse stiffness* K_T of $\mathscr{B}$

$$\mathsf{K}_T := \frac{T}{4AB\mu} \left(\frac{s}{L} \right)^{-1}. \tag{62}$$

The comparison of (62) with (61) shows that the stiffness K_T depends on the normal load

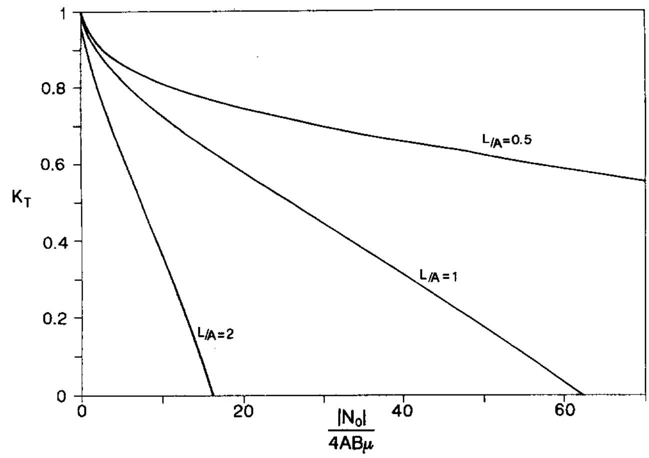

Fig. 4. Transverse stiffness K_T as a function of $|N_0|/4AB\mu$ ($L/A = 0.5, 1, 2$; $\sigma = 20$, $\lambda/\mu = 0.8$).

N_0. In particular, for a number of fixed values of L/A, Fig. 4 illustrates the behavior of K_T in terms of the dimensionless average pressure $|N_0|/4AB\mu$. Interestingly, K_T is a decreasing function of $|N_0|$; it means that, under shear force, the compressed block is easier to deform for increasing normal force $|N_0|$. Further, by (61) and (62) we see that K_T vanishes at the value, k_{cr} say, such that

$$d(k_{cr})L = \pi. \tag{63}$$

We call $N_0(k_{cr})$ the critical load for small shear deformations superimposed on finite compression. The curve (b) in Fig. 3 illustrates the dependence of such a critical load on the slenderness L/A.

4.3. *Superimposed small bending*

In this case the boundary conditions have the form

$$\bar{g}(0) = 0,\ \bar{h}(0) = 0,\ \bar{v}(0) = 0 \quad \text{and} \quad \bar{g}(L) = 0,\ \bar{h}(L) = 0,\ \bar{v}(L) = -\phi; \tag{64}$$

that is, the basis $X_3 = 0$ is fixed and the other basis undergoes a rotation of amount ϕ about the X_2-axis. The consequent boundary-value problem (47), (64) can be solved in a way similar to that considered in the previous subsection. In particular, we obtain

$$\begin{aligned}
C_1 &= C_2 = 0,\\
C_3 &= -\frac{\phi}{2V - V^2 kL\cot\dfrac{d(k)L}{2}},\\
C_4 &= -C_6 = \frac{VkL - \sin(d(k)L)}{2V\sin^2\dfrac{d(k)L}{2}\left(2 - VkL\cot\dfrac{d(k)L}{2}\right)}\phi,\\
C_5 &= \frac{\phi}{2 - VkL\cot\dfrac{d(k)L}{2}},
\end{aligned} \tag{65}$$

where we have set

$$V = \frac{b(k)d(k)}{c(k)}. \tag{66}$$

We conclude by recording a significant formula which we derive from the scrutiny of the current problem. If we call

$$m := \mathbf{a}_2 \cdot \int_{\mathscr{A}(L)} (\mathbf{x} - \mathbf{f}(0,0,L)) \times \mathbf{S}\mathbf{e}_3 \, \mathrm{d}A \tag{67}$$

the bending moment acting on the end face $\mathscr{A}(L)$, we have from (27), (21) and (34) that

$$m = M(L) - h(L)T(L) + g(L)N(L). \tag{68}$$

Thus, the following expression for m

$$m = \frac{A^2}{6} a(k)d(k) \frac{\sin(d(k)L) - Vkl\cos(d(k)L)}{\sin^2 \frac{d(k)L}{2}\left(2 - VkL\cot\frac{d(k)L}{2}\right)} \phi \tag{69}$$

follows from (43), (48), (65) and (66). In Fig. 5, for a number of fixed values of L/A, we plot the dimensionless *bending stiffness* K_M of $\mathscr{B}$, defined by

$$\mathsf{K}_\mathsf{M} := \frac{m}{A^2 B\mu} \frac{1}{\phi}, \tag{70}$$

against the average pressure $|N_0|/4AB\mu$. It is worth noting that, for values of L/A chosen as in the present case, the bending stiffness K_M goes through a maximum as the normal load $|N_0|$ is increased. Further, by (69) and (66) it is easily seen that K_M vanishes when the numerator in (69) is zero. Let k_{cr} denote the unique value of k in $]k^1_{cr}, 1[$, with k^1_{cr} defined by (54), for which this condition is verified. Then, curve (c) in Fig. 3 illustrates the relation between the corresponding critical load $N_0(k_{cr})$ and the slenderness L/A.

5. COMPARISON WITH PREVIOUS ANALYSIS AND EXPERIMENTS

Our theoretical investigation has specific applications in predicting the mechanical behavior of multilayered bearings. In order to illustrate this point, we now comment on the

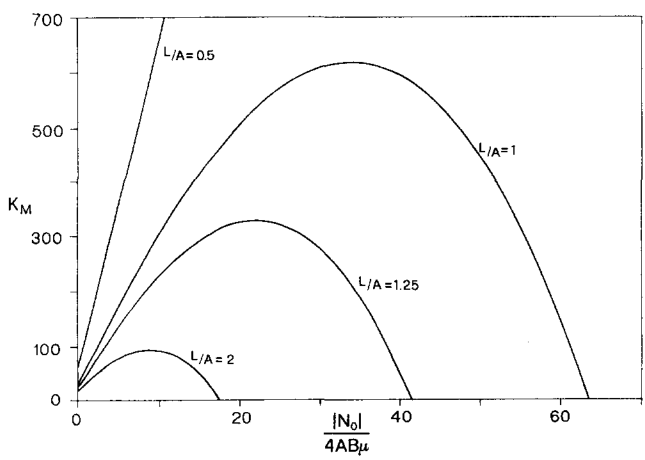

Fig. 5. Bending stiffness K_M as a function of $|N_0|/4AB\mu$ ($L/A = 0.5, 1, 1.25, 2$; $\sigma = 20$, $\lambda/\mu = 0.8$).

significance of our results by discussing some comparisons with previous analysis and experiments.

We begin our discussion, which will be limited to the stability, with a remark on our reasons for studying the small deformations superimposed on a finite compression (Section 4). Such a study has first of all primary interest as a model of the mechanical behavior of solids and structures; secondly, it allows one to formulate the stability, namely, as observed by Truesdell and Toupin (1960), as an application of the theory of finite deformation. Thirdly, as we will see to be confirmed by experiments, superimposed deformations seem appropriate to capture some significant non-linear effects in the structural response of rubber bearings. We also notice that the stability analysis has a fundamental role in the design of reinforced elastomeric bearings. However, the specifications of usual codes, which limit the slenderness of the bearing to prevent buckling phenomena, are generally very conservative; so, bearings are used which are stiffer than necessary with respect to horizontal displacements.

Past theoretical work on the stability of multilayered rubber bearings is essentially based on linear approaches. Here we will consider two of such approaches, both including shear flexibility; precisely, the so-called Haringx's theory and its subsequent modification proposed by Gent.

For convenience of the reader, we next recall the main results of such theories; greater details can be found in Buckle and Kelly (1986) and Stanton *et al.* (1990). Consider an axially loaded bearing that is fixed against the rotation at both ends. Let T and s denote the shear force and the relative displacement of the bases, respectively. Thus, the *transverse stiffness* T/s of the bearing is given by the formula:

$$\frac{T}{s} = \frac{\dfrac{P_0}{L}}{2\dfrac{1+P_0/\mathscr{K}_s}{qL}\tan\dfrac{qL}{2}-1}, \tag{71}$$

where

$$q^2 = \frac{P_0}{\mathscr{K}_r}\left(1+\frac{P_0}{\mathscr{K}_s}\right), \tag{72}$$

and where P_0 is the compressive axial load, L is the height of the bearing, $\mathscr{K}_r$ and $\mathscr{K}_s$ are the flexural and shear stiffnesses, respectively. Again in the context of the above-mentioned models, the *lowest critical value* P_{cr} of the axial load is expressed by

$$P_{cr} = \frac{\mathscr{K}_s}{2}\left[-1+\sqrt{1+\frac{16\pi^2\mathscr{K}_r}{\mathscr{K}_s L^2}}\right], \tag{73}$$

when the bases of the bearing are assumed both fixed.

As regards $\mathscr{K}_r$ and $\mathscr{K}_s$, Haringx's model adopts the corresponding expressions which hold in the engineering beam theory. So that, for a bearing whose cross section is a square with edge length $2A$, we have

$$\mathscr{K}_r = \frac{1}{12}(2A)^4E, \quad \mathscr{K}_s = 4A^2G, \tag{74}$$

if the rubber has Young's modulus E and shear modulus G. On the contrary, in Gent's approach $\mathscr{K}_r$ and $\mathscr{K}_s$ are derived from the properties of a typical rubber layer, and then adjusted to account for the presence of the steel layers. In this case, for the same bearing considered in (74), one has

$$\mathscr{K}_r = \frac{L}{L_r} f_r \frac{1}{12}(2A)^4 E, \quad \mathscr{K}_s = \frac{L}{L_r} 4A^2 G, \tag{75}$$

where L_r is the total thickness of rubber, and f_r is a numerical coefficient depending on the *shape factor* of the typical rubber layer. If t_r denotes the thickness of the typical rubber layer, then f_r is given by

$$f_r = 1 + 0.1855\left(\frac{A}{t_r}\right)^2. \tag{76}$$

As noted by Stanton *et al.* (1990), Gent also performed experiments on rubber bearings. Only for very slender bearings did he find a reasonable agreement with the predictions of his theoretical model. Indeed, for bearings with practical dimensions, the agreement was neither clear nor remarkable. Finally, the experiments performed by Buckle and Kelly (1986) on seismic isolation bearings showed the validity of the improvement proposed by Gent to Haringx's model.

We now compare the results just quoted with our results on the lowest critical load in simple compression (Subsection 4.1) and the transverse stiffness (Subsection 4.2). Clearly, to unify the different notations, we have to set the obvious identifications:

$$P_0 = -N_0 \quad \text{and} \quad P_{cr} = -N_0(k^1_{cr}); \tag{77}$$

here, P_0 and P_{cr} are the notation used in (71), (72) and (73), while our N_0 and $N_0(k^1_{cr})$ are defined by (41) and (44), (54), respectively. Further, in view of (74), (75), (8) and (17), we assume the following relations for the material moduli:

$$G = \frac{E}{3} = \frac{L_r}{L}\mu, \quad \frac{\lambda}{\mu} = 0.8, \quad \sigma = 10. \tag{78}$$

Finally, as regards the geometrical characteristics of the rubber layers, we consider the practical situation (Buckle and Kelly, 1986) defined by the conditions:

$$\frac{A}{t_r} = 15.1, \quad \frac{L_r}{L} = 0.75. \tag{79}$$

Figure 6 shows the relation between the dimensionless buckling load $|N_0(k^1_{cr})|/4A^2\mu$

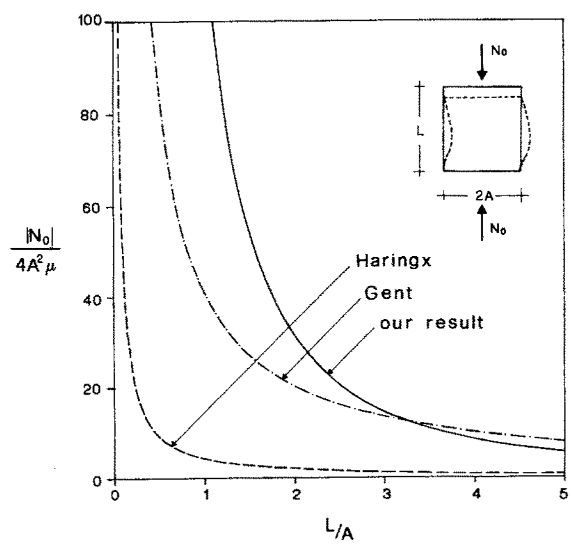

Fig. 6. Buckling load $|N_0(k^1_{cr})|/4A^2\mu$ as a function of slenderness L/A, in Haringx's theory, Gent's modification and our model.

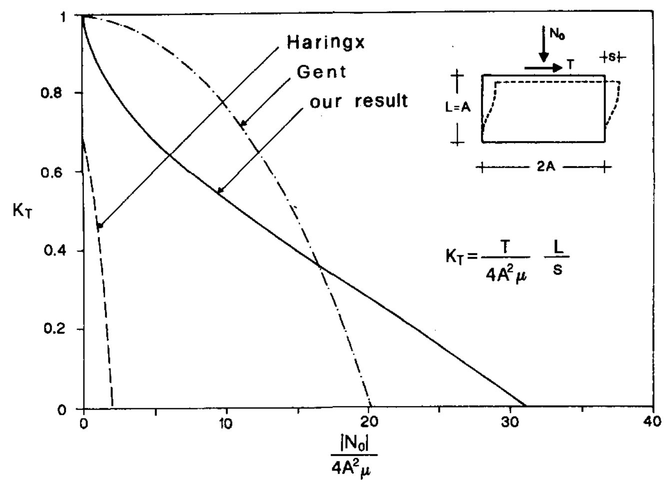

Fig. 7. Transverse stiffness K_T as a function of the axial load $|N_0|/4A^2\mu$, in Haringx's theory, Gent's modification and our model.

and the slenderness L/A, as predicted by Haringx's theory [eqns (73), (74)], Gent's modification [eqns (73), (75)] and our model [eqns (44), (54)]. Figure 7 illustrates an analogous comparison for the transverse stiffness as a function of the axial load $|N_0|/4A^2\mu$; Haringx's curve is deduced from (71), (72) and (74), Gent's curve from (71), (72) and (75), and our curve is derived from eqn (61). All curves in Fig. 7 are obtained by assuming $L/A = 1$.

To analyze the predictions shown in the above figures, the interesting discussion of Stanton *et al.* (1990) may be useful. Indeed, these researchers discussed the experiments conducted on columns made from steel-reinforced bearings to determine the buckling load and the transverse stiffness. First of all, we remark that Gent's theory improves Haringx's, and this experimental fact is clearly confirmed by the predictions described in both Figs 6 and 7, where Gent's curves are near enough to our curves. Further, as already said, Gent's model gives good predictions for the buckling load of multilayered bearings that are very slender (hence, at very low stresses), even if it slightly overpredicts the experimental values. For bearings of practical dimensions (i.e. for stock slendernesses), Gent's model considerably underpredicts the experimental buckling loads. Because of such a deficiency, as noted by Stanton *et al.* (1990), the model proposed by Gent must be modified to account for changes in geometry, which are significant at high levels of stress. All of the above observations are quite in agreement with the predictions illustrated by our curve, as can be seen in Fig. 6.

Tests were also performed on compressed bearings to measure the transverse stiffness under varying amounts of axial load (Stanton *et al.*, 1990). In these tests, the transverse stiffness was taken as the *tangent* stiffness at zero displacement, as in our corresponding analysis of small shear superimposed on finite compression. The results of all tests indicate that the transverse stiffness decreases with increasing axial load, and this agrees with the predictions shown in Fig. 7. However, the reduction in transverse stiffness shown by experiments was greater than Gent's predictions (compare with our curve). Moreover, if changes in geometry are taken into account, a curve is deduced by experimental results (Stanton *et al.*, 1990, p. 1361, Fig. 4b), which seems to be in remarkable agreement with our curve in Fig. 7.

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